WEAK TYPE COMMUTATOR AND LIPSCHITZ ESTIMATES: RESOLUTION OF THE NAZAROV-PELLER CONJECTURE

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ABSTRACT. Let \mathcal{M} be a semi-finite von Neumann algebra and let $f: \mathbb{R} \to \mathbb{C}$ be a Lipschitz function. If $A, B \in \mathcal{M}$ are self-adjoint operators such that $[A, B] \in L_1(\mathcal{M})$, then

$$||[f(A), B]||_{1,\infty} \le c_{abs} ||f'||_{\infty} ||[A, B]||_{1},$$

where c_{abs} is an absolute constant independent of f, \mathcal{M} and A, B and $\|\cdot\|_{1,\infty}$ denotes the weak L_1 -norm. If $X, Y \in \mathcal{M}$ are self-adjoint operators such that $X - Y \in L_1(\mathcal{M})$, then

$$||f(X) - f(Y)||_{1,\infty} \le c_{abs} ||f'||_{\infty} ||X - Y||_1.$$

This result resolves a conjecture raised by F. Nazarov and V. Peller implying a couple of existing results in perturbation theory.

1. Introduction

Let $L_p(H)$ be the Schatten-von Neumann ideal of B(H). It consists of all compact operators for which its sequence of singular values lies in ℓ_p . Let F_p be the class of functions $f: \mathbb{R} \to \mathbb{C}$ such that

$$f(B) - f(C) \in L_p(H),$$

for all self-adjoint B, C such that $B - C \in L_p(H)$ and set

$$||f||_{F_p} = \sup_{B \neq C} \frac{||f(B) - f(C)||_p}{||B - C||_p}.$$

It was conjectured by M.G. Krein [17] that whenever the derivative $f' \in L_{\infty}(\mathbb{R})$ we have $f \in F_1$. This conjecture does not hold as was shown by Y.B. Farforovskaya in [11]. Also it was shown that the analogue of Krein's problem fails in the case $p = \infty$ (see [9, 10]). In fact already for the absolute value function it was found by T. Kato that Krein's problem has a negative answer [15]; and similarly in the case p = 1 by E.B. Davies [5].

A positive result in this direction was first obtained by M. Birman and M. Solomyak [1, Theorem 10] who proved that $C^{1+\epsilon}\subseteq F_1$ for every $\epsilon>0$, and later improved by V. Peller [22] who showed that $B^1_{\infty 1}\subseteq F_1$. Here B^s_{pq} is the class of Besov spaces for which we refer to [12]. The Krein problem for the case $1 remained open until [24]. In [24] it was shown by the second and third named author that <math>F_p$ consists exactly of all Lipschitz functions. Moreover in [3] a quantitative estimate for $||f||_{F_p}$ was found, namely $||f||_{F_p} \simeq p^2/(p-1)$. Earlier the same problem had been considered by M. de la Salle (unpublished, see

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[27]) who was able to show already that $||f||_{F_p} \le c_{abs} \cdot p^4/(p-1)^2$ with an absolute constant c_{abs} , which is more optimal than [24].

Other results concerning this problem have been obtained in [8] and [16] and in the context of this paper we also mention [26] in which weak estimates for martingale inequalities were obtained.

Using interpolation, the above results would follow from a weak type Lipschitz estimate between L_1 and the weak- L_1 space $L_{1,\infty}$. The estimate was conjectured in a paper of F. Nazarov and V. Peller [19], and has remained the major open question in the study of Lipschitz properties of operator valued functions. Denote $L_{1,\infty}(H)$ for the weak L_1 -space consisting of all compact operators A whose sequence $\{\mu(k,A)\}_{k\geq 0}$ of singular values satisfies $\mu(k,A) = O(\frac{1}{k+1})$.

Conjecture 1.1. Let $f: \mathbb{R} \to \mathbb{C}$ be Lipschitz. Whenever $A, B \in B(H)$ are self-adjoint operators such that $A - B \in L_1(H)$, we have that $f(A) - f(B) \in L_{1,\infty}(H)$ and

$$||f(A) - f(B)||_{1,\infty} \le c_{abs} ||f'||_{\infty} ||A - B||_1,$$

for some absolute constant c_{abs} .

Nazarov and Peller [19] gave an affirmative answer under the assumption that the rank of A-B equals 1. Since $\|\cdot\|_{1,\infty}$ is a quasi-norm and not a norm for $L_{1,\infty}(H)$ it is impossible to extend their result for when A-B is a general trace class operator.

Another positive result to the conjecture was found by the current authors in [4] in the special case when f is the absolute value map. The proof relies on the observation that the Schur multiplier of divided differences

$$\left(\frac{f(\lambda) - f(\mu)}{\lambda - \mu}\right)_{\lambda \neq \mu}$$

can be written as a finite sum of compositions of a positive definite Schur multiplier and a triangular truncation operator. For general Lipschitz functions there is no reason that the latter fact should be true which renders the technique of [4] inapplicable.

The main result of this paper is a proof of Conjecture 1.1. The importance of this result lies in the fact that this gives the sharpest possible estimate for perturbations and commutators. In particular it retrieves $||f||_{F_p} \simeq p^2/(p-1)$ [3] and the Nazarov–Peller result [19]. A key ingredient in our proof is the connection with non-commutative Calderón–Zygmund theory and in particular J. Parcet's extension of the classical Calderón–Zygmund theorem (see Theorem 2.1 and [21]).

In the text we prove a somewhat stronger result in the terms of double operator integrals (see next section for definition), of which Conjecture 1.1 is a corollary.

Theorem 1.2. If A is a self-adjoint operator affiliated with a semifinite von Neumann algebra \mathcal{M} , and if $f: \mathbb{R} \to \mathbb{C}$ is Lipschitz then

$$||T_{f^{[1]}}^{A,A}(V)||_{1,\infty} \le c_{abs}||f'||_{\infty}||V||_{1}, \quad V \in (L_{1} \cap L_{2})(\mathcal{M}).$$

Commutator estimate follows from the observation that the double operator integral $T_{f^{[1]}}^{A,A}([A,B])$ equals [f(A),B]. As explained in the proof of Theorem 5.3, Lipschitz estimates follow from commutator ones.

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2. Preliminaries

2.1. **General notation.** Let \mathcal{M} be a semifinite von Neumann algebra equipped with a faithful normal semifinite trace τ . In this paper, we always presume that \mathcal{M} is represented on a separable Hilbert space.

A (closed and densely defined) operator x affiliated to \mathcal{M} is called τ -measurable if $\tau(E_{|x|}(s,\infty)) < \infty$ for sufficiently large s. We denote the set of all τ -measurable operators by $S(\mathcal{M},\tau)$. For every $x \in S(\mathcal{M},\tau)$, we define its singular value function $\mu(A)$ by setting

$$\mu(t,x) = \inf\{\|x(1-p)\|_{\infty} : \tau(p) \le t\}.$$

Equivalently, for positive operator $x \in S(\mathcal{M}, \tau)$, we have

$$n_x(s) = \tau(E_x(s, \infty)), \quad \mu(t, x) = \inf\{s : n_x(s) < t\}.$$

We have (see e.g. [18, Corollary 2.3.16])

(2.1)
$$\mu(t+s, x+y) \le \mu(t, x) + \mu(s, y), \quad t, s > 0.$$

2.2. Non-commutative spaces. For $1 \le p < \infty$ we set,

$$L_p(\mathcal{M}) = \{ x \in S(\mathcal{M}, \tau) : \ \tau(|x|^p) < \infty \}, \quad ||x||_p = (\tau(|x|^p))^{\frac{1}{p}}.$$

The Banach spaces $(L_p(\mathcal{M}), \|\cdot\|_p)$, $1 \leq p < \infty$ are separable.

Define the space $L_{1,\infty}(\mathcal{M})$ by setting

$$L_{1,\infty}(\mathcal{M}) = \{ x \in S(\mathcal{M}, \tau) : \sup_{t>0} t\mu(t, x) < \infty \}.$$

We equip $L_{1,\infty}(\mathcal{M})$ with the functional $\|\cdot\|_{1,\infty}$ defined by the formula

$$||x||_{1,\infty} = \sup_{t>0} t\mu(t,x), \quad x \in L_{1,\infty}(\mathcal{M}).$$

It follows from (2.1) that

$$||x+y||_{1,\infty} = \sup_{t>0} t\mu(t,x+y) \le \sup_{t>0} t(\mu(\frac{t}{2},x) + \mu(\frac{t}{2},y)) \le$$
$$\le \sup_{t>0} t\mu(\frac{t}{2},x) + \sup_{t>0} t\mu(\frac{t}{2},y) = 2||x||_{1,\infty} + 2||y||_{1,\infty}.$$

In particular, $\|\cdot\|_{1,\infty}$ is a quasi-norm. The quasi-normed space $(L_{1,\infty}(\mathcal{M}), \|\cdot\|_{1,\infty})$ is, in fact, quasi-Banach (see e.g. [14, Section 7] or [30]). In view of our main result it is important to emphasize that the quasi-norm $\|\cdot\|_{1,\infty}$ is not equivalent to any norm on $L_{1,\infty}(\mathcal{M})$ (see e.g [14, Theorem 7.6]).

2.3. Weak type inequalities for Calderón-Zygmund operators. Parcet [21] proved a noncommutative extension of Calderón-Zygmund theory.

Let K be a tempered distribution which we refer to as the *convolution kernel*. We let W_K be the associated Calderón-Zygmund operator, formally given by $f \mapsto K * f$. In what follows, we only consider tempered distributions having local values (that is, which can be identified with measurable functions $K : \mathbb{R}^d \to \mathbb{C}$).

Let \mathcal{M} be a semi-finite von Neumann algebra with normal, semi-finite, faithful trace τ . The operator $1 \otimes W_K$ can, under suitable conditions, be defined as noncommutative Calderón-Zygmund operators by letting them act on the second tensor leg

of $L_1(\mathcal{M}) \widehat{\otimes} L_1(\mathbb{R}^d)$. The following theorem in particular gives a sufficient condition for such an operator to act from L_1 to $L_{1,\infty}$.

Theorem 2.1 ([21]). Let $K : \mathbb{R}^d \setminus \{0\} \to \mathbb{C}$ be a kernel satisfying the conditions¹

$$(2.2) |K|(t) \le \frac{\text{const}}{|t|^d}, |\nabla K|(t) \le \frac{\text{const}}{|t|^{d+1}}.$$

Let \mathcal{M} be a semi-finite von Neumann algebra. If $W_K \in B(L_2(\mathbb{R}^d))$, then the operator $1 \otimes W_K$ defines a bounded map from $L_1(\mathcal{M} \otimes L_{\infty}(\mathbb{R}^d))$ to $L_{1,\infty}(\mathcal{M} \otimes L_{\infty}(\mathbb{R}^d))$.

2.4. Double operator integrals. Let $A = A^*$ be an operator affiliated with \mathcal{M} . Symbolically, a double operator integral is defined by the formula

(2.3)
$$T_{\xi}^{A,A}(V) = \int_{\mathbb{R}^2} \xi(\lambda,\mu) dE_A(\lambda) V E_A(\mu), \quad V \in L_2(\mathcal{M}).$$

In the subsequent paragraph, we provide a rigorous definition of the double operator integral.

Consider projection valued measures on \mathbb{R} acting on the Hilbert space $L_2(\mathcal{M})$ by the formulae $x \to E_A(\mathcal{B})x$ and $x \to xE_A(\mathcal{B})$. These spectral measures commute and, hence (see Theorem V.2.6 in [2]), there exists a countably additive (in the strong operator topology) projection-valued measure ν on \mathbb{R}^2 acting on the Hilbert space $L_2(\mathcal{M})$ by the formula

(2.4)
$$\nu(\mathcal{B}_1 \otimes \mathcal{B}_2) : x \to E_A(\mathcal{B}_1) x E_A(\mathcal{B}_2), \quad x \in L_2(\mathcal{M}).$$

Integrating a bounded Borel function ξ on \mathbb{R}^2 with respect to the measure ν produces a bounded operator acting on the Hilbert space $L_2(\mathcal{M})$. In what follows, we denote the latter operator by $T_{\xi}^{A,A}$ (see also [20, Remark 3.1]). In the special case when A is bounded and $\operatorname{spec}(A) \subset \mathbb{Z}$, we have

(2.5)
$$T_{\xi}^{A,A}(V) = \sum_{k,l \in \mathbb{Z}} \xi(k,l) E_A(\{k\}) V E_A(\{l\}).$$

We are mostly interested in the case $\xi = f^{[1]}$ for a Lipschitz function f. Here,

$$f^{[1]}(\lambda,\mu) = \begin{cases} \frac{f(\lambda) - f(\mu)}{\lambda - \mu}, & \lambda \neq \mu \\ 0, & \lambda = \mu. \end{cases}$$

3. Approximate intertwining properties of Fourier multipliers

We prove intertwining properties of Fourier multipliers partly inspired by K. de Leeuw's proof of his restriction theorem for L^p -multipliers [6, Section 2].

$$G_l(s) = \frac{1}{l\sqrt{\pi}}e^{-\left(\frac{s}{l}\right)^2}, \quad s \in \mathbb{R}, \quad l > 0.$$

That is, G_l is a probability density function for certain Gaussian random variable. The notation $G_l^{\otimes d}$ stands for the function from $L_1(\mathbb{R}^d)$ given by the tensor product of G_l with itself repeated d times.

Lemma 3.1. For every $f \in L_1(\mathbb{R})$ with $\int_{-\infty}^{\infty} f(s)ds = 0$, we have $f * G_l \to 0$ in $L_1(\mathbb{R})$ as $l \to \infty$.

¹Here, ∇ denotes the gradient $(\frac{1}{i}\frac{\partial}{\partial x_1}, \cdots, \frac{1}{i}\frac{\partial}{\partial x_d})$, which is understood as unbounded selfadjoint operator on $L_2(\mathbb{R}^d)$.

Proof. Suppose first that f is a step function of the form $f = \sum_{k=1}^{m} \alpha_k \chi_{I_k}, m \ge 1$ where $I_k = [a_k, b_k], \ 1 \le k \le m$ are disjoint intervals and $\sum_{k=1}^{m} \alpha_k m(I_k) = 0$. We have

$$(f * G_l)(t) = \sum_{k=1}^{m} \alpha_k \int_{a_k}^{b_k} G_l(t-s) ds = \sum_{k=1}^{m} \alpha_k \int_{t-b_k}^{t-a_k} G_l(u) du = \sum_{k=1}^{m} \alpha_k \int_{\frac{t-a_k}{l}}^{\frac{t-a_k}{l}} G_1(s) ds = \sum_{k=1}^{m} \alpha_k (F(\frac{t-a_k}{l}) - F(\frac{t-b_k}{l})),$$

where $F(t) = \int_{-\infty}^{t} G_1(s)ds$. To prove the assertion for our f, it suffices to show that

$$l \int_{-\infty}^{\infty} \left| \sum_{k=1}^{m} \alpha_k \left(F\left(t - \frac{a_k}{l}\right) - F\left(t - \frac{b_k}{l}\right) \right) \right| dt \to 0$$

Clearly,

$$\left| F(t - \frac{a_k}{l}) - F(t) + \frac{a_k}{l} F'(t) \right| \le \frac{a_k^2}{2l^2} \max_{s \in [t - \frac{a_k}{l}, t]} |F''(s)|.$$

If $l > \max_{1 \le k \le m} |a_k|$ and $l > \max_{1 \le k \le m} |b_k|$, then

$$\left| \sum_{k=1}^{m} \alpha_k (F(t - \frac{a_k}{l}) - F(t - \frac{b_k}{l})) \right| \le \frac{1}{2l^2} (\sum_{k=1}^{m} |\alpha_k| (a_k^2 + b_k^2)) \max_{s \in [t-1, t+1]} |F''(s)|.$$

This proves the assertion for f as above.

To prove the assertion in general, fix f_m as above (i.e., mean zero step functions) such that $f_m \to f$ in $L_1(\mathbb{R})$. Since $||G_l||_1 = 1$, it follows from Young's inequality that

$$||f * G_l||_1 \le ||(f - f_m) * G_l||_1 + ||f_m * G_l||_1 \le ||f - f_m||_1 + ||f_m * G_l||_1.$$

Therefore,

$$\limsup_{l \to \infty} \|f * G_l\|_1 \le \|f - f_m\|_1.$$

Passing $m \to \infty$, we conclude the proof.

By Fubini Theorem, linear span of elementary tensors

$$(f_1 \otimes \cdots \otimes f_d) : (t_1, \cdots, t_d) \to f_1(t_1) \cdots f_d(t_d), \quad f_1, \cdots, f_d \in L_1(\mathbb{R})$$

is dense in $L_1(\mathbb{R}^d)$.

Lemma 3.2. For every $f \in L_1(\mathbb{R}^d)$ with $\int_{\mathbb{R}^d} f(s)ds = 0$, we have $f * G_l^{\otimes d} \to 0$ in $L_1(\mathbb{R}^d)$ as $l \to \infty$.

Proof. Suppose first that f is a linear combination of elementary tensors. That is,

(3.1)
$$f = \sum_{k=1}^{m} \bigotimes_{j=1}^{d} f_{jk}, \quad f_{jk} \in L_1(\mathbb{R}).$$

Firstly, we consider the case when for every $k, 1 \le k \le m$ there exists $j, 1 \le j \le d$ such that $\int_{\mathbb{R}} f_{jk}(s) = 0$. In this case, by Lemma 3.1 we have that

$$||f * G_l^{\otimes d}||_1 \le \sum_{k=1}^m \prod_{i=1}^d ||f_{jk} * G_l||_1 \to 0.$$

Now, we show that the case of f given by (3.1) satisfying

(3.2)
$$\sum_{k=1}^{m} \prod_{j=1}^{d} \int_{\mathbb{R}} f_{jk}(s) ds = 0.$$

can be reduced to the just considered case when for every $k, 1 \le k \le m$ there exists $j, 1 \le j \le d$ such that $\int_{\mathbb{R}} f_{jk}(s) = 0$. To this end, for every subset $\mathscr{A} \subset \{1, \dots, d\}$, we set

$$f_{j,k,\mathscr{A}} = \begin{cases} f_{jk} - (\int_{\mathbb{R}} f_{jk}(s)ds)\chi_{(0,1)}, & j \in \mathscr{A} \\ (\int_{\mathbb{R}} f_{jk}(s)ds)\chi_{(0,1)}, & j \notin \mathscr{A}. \end{cases}$$

By the linearity, we can rewrite (3.1) as

$$f = \sum_{k=1}^{m} \sum_{\mathscr{A} \subset \{1, \dots, d\}} \bigoplus_{j=1}^{d} f_{j,k,\mathscr{A}}$$

Observing now that

$$\sum_{k=1}^{m} \bigotimes_{j=1}^{d} f_{j,k,\varnothing} = (\sum_{k=1}^{m} \prod_{j=1}^{d} \int_{\mathbb{R}} f_{jk}(s) ds) \chi_{(0,1)}^{\otimes d}$$

and appealing to (3.2), we arrive at

(3.3)
$$f = \sum_{k=1}^{m} \sum_{\varnothing \neq \varnothing \subset \{1, \dots, d\}} \bigotimes_{j=1}^{d} f_{j,k,\varnothing}.$$

Note that $f_{j,k,\mathscr{A}}$ is mean zero for $j \in \mathscr{A}$. Using representation (3.3) instead of (3.1) for f, we may assume without loss of generality that for every k, $1 \leq k \leq m$ there exists j, $1 \leq j \leq d$ such that $\int_{\mathbb{R}} f_{jk}(s) = 0$. This completes the proof of the lemma in the special case when f is given by (3.1) and satisfies (3.2).

To prove the general case, fix $f \in L_1(\mathbb{R}^d)$ with $\int_{\mathbb{R}^d} f(s)ds = 0$, and select a sequence $\{f_m\}_{m=1}^{\infty}$ of mean zero sums of elementary tensors such that $f_m \to f$ in $L_1(\mathbb{R}^d)$ as $m \to \infty$. Since $\|G_l^{\otimes d}\|_1 = 1$, $l \ge 1$ it follows from Young inequality that

$$||f * G_l^{\otimes d}||_1 \le ||(f - f_m) * G_l^{\otimes d}||_1 + ||f_m * G_l^{\otimes d}||_1 \le ||f - f_m||_1 + ||f_m * G_l^{\otimes d}||_1.$$

Therefore,

$$\limsup_{l \to \infty} \|f * G_l^{\otimes d}\|_1 \le \|f - f_m\|_1.$$

Passing $m \to \infty$, we conclude the proof.

In what follows,

(3.4)
$$e_k(t) := e^{i\langle k, t \rangle}, \quad k, t \in \mathbb{R}^d$$

and \mathcal{F} stands for the Fourier transform.

Lemma 3.3. If $g \in L_{\infty}(\mathbb{R}^d)$ is such that $\mathcal{F}(g) \in L_1(\mathbb{R}^d)$, then for every $k \in \mathbb{R}^d$ we have

$$(g(\nabla))(G_l^{\otimes d}e_k) - g(k)G_l^{\otimes d}e_k \to 0$$

in $L_1(\mathbb{R}^d)$ as $l \to \infty$.

Proof. Fix $k \in \mathbb{R}^d$. Set $h_1(t) := g(k)e^{-|t-k|^2}$ and $h_0(t) := g(t) - h_1(t)$, $t \in \mathbb{R}^d$. Observe that, for every $t \in \mathbb{R}^d$, we have

$$\mathcal{F}(G_l^{\otimes d})(t) = \pi^{-d/2} e^{-l^2|t|^2}$$

Since $h_1(\nabla)$ on the Fourier side is a multiplier on h_1 , it follows that, for every $t \in \mathbb{R}^d$,

$$\mathcal{F}(G_l^{\otimes d}e_k)(t) = \pi^{-d/2}e^{-l^2|t-k|^2}, \quad (\mathcal{F}((h_1(\nabla))(G_l^{\otimes d}e_k)))(t) = g(k)\pi^{-d/2}e^{-(l^2+1)|t-k|^2}.$$

Applying the inverse Fourier transform to the second equality, we arrive at

$$(h_1(\nabla))(G_l^{\otimes d}e_k) = g(k)G_{(l^2+1)^{1/2}}^{\otimes d}e_k.$$

A direct computation yields $G_{(l^2+1)^{1/2}}^{\otimes d} - G_l^{\otimes d} \to 0$ in $L_1(\mathbb{R}^d)$ as $l \to \infty$. We conclude that

$$(h_1(\nabla))(G_l^{\otimes d}e_k) - g(k)G_l^{\otimes d}e_k \to 0$$

in $L_1(\mathbb{R}^d)$ as $l \to \infty$. It, therefore, suffices to show that

$$(h_0(\nabla))(G_l^{\otimes d}e_k) \to 0$$

in $L_1(\mathbb{R}^d)$ as $l \to \infty$. Define the function $f \in L_1(\mathbb{R}^d)$ by setting $f(t) = e^{i\langle k, t \rangle}(\mathcal{F}h_0)(t)$, $t \in \mathbb{R}^d$. We rewrite the latter equation as $f * G_l^{\otimes d} \to 0$ as $l \to \infty$. Note that

$$\int_{\mathbb{R}^d} f(s)ds = \int_{\mathbb{R}^d} e^{i\langle k, s \rangle} (\mathcal{F}h_0)(s)ds = h_0(k) = 0.$$

The assertion follows now from the Lemma 3.2.

4. Proof of Theorem 1.2 in the special case

For s > 0, the dilation operator σ_s acts on the space of Lebesgue measurable functions on \mathbb{R} , by the formula $(\sigma_s x)(t) = x(t/s)$.

Lemma 4.1. Let x, y be measurable and θ be integrable functions on \mathbb{R} . Let $z(t) := t^{-1}$, t > 0, z(t) = 0, t < 0, and let u > 0. For Lebesgue measurable functions $x \otimes y$ and $\theta \otimes z$ on \mathbb{R}^2 , we have

$$\mu(\sigma_u(x) \otimes y) = \sigma_u \mu(x \otimes y), \quad \mu(t, \theta \otimes z) = \|\theta\|_1 t^{-1}, \quad t > 0.$$

Proof. Denoting Lebesgue measure on \mathbb{R}^2 by m, we have for every t>0

$$m(\{\sigma_u(x) \otimes y > t\}) = m(\{(s_1, s_2) : x(\frac{s_1}{u})y(s_2) > t\})$$

= $um(\{(s_1, s_2) : x(s_1)y(s_2) > t\})$
= $um(\{x \otimes y > t\}).$

This proves the first assertion.

Firstly, we prove the second assertion for simple function $x \in L_1(\mathbb{R})$. If $x = \sum_k a_k \chi_{B_k}$ with B_k being pairwise disjoint sets, then²

$$\mu(x \otimes z) = \mu(\bigoplus_k (a_k \chi_{B_k} \otimes z)) = \mu(\bigoplus_k \mu((a_k \chi_{B_k} \otimes z)).$$

²The notation $\bigoplus_k x_k$ stands for disjoint sum of the functions x_k , that is $\sum_k z_k$, where functions z_k have pairwise disjoint support and $\mu(z_k) = \mu(x_k)$. We refer the reader to the Definition 2.4.3 in [18] and subsequent comments.

If B is a set of finite measure, then there exists a measure preserving bijection from B to (0, m(B)) (see [13]). Therefore, we have

$$\mu(\chi_B \otimes z) = \mu(\chi_{(0,m(B))} \otimes z) = m(B)z.$$

Thus,

$$\mu(x \otimes z) = \mu(\bigoplus_{k} |a_k| m(B_k)z) = (\sum_{k} a_k m(B_k))z.$$

The second assertion follows now by approximation

Lemma 4.2. For every $X \in L_{1,\infty}(\mathcal{M})$ and every l > 0, we have

$$e^{-d}\pi^{-\frac{d}{2}}\|X\|_{1,\infty} \le \|X \otimes G_l^{\otimes d}\|_{1,\infty} \le \|X\|_{1,\infty}.$$

Proof. For every operator $A \in S(\mathcal{M}, \tau)$ and for every function $g \in L_{\infty}(0, \infty)$, we have³

Let z be as in Lemma 4.1. It follows from the definition $\|\cdot\|_{1,\infty}$ that $\mu(X) \leq \|X\|_{1,\infty}z$ and, therefore,

$$\mu(X \otimes G_l^{\otimes d}) \stackrel{(4.1)}{=} \mu(\mu(X) \otimes G_l^{\otimes d}) \le \|X\|_{1,\infty} \mu(z \otimes G_l^{\otimes d}) \stackrel{L.4.1}{=} \|X\|_{1,\infty} \mu(z).$$

This proves the right hand side inequality.

On the other hand, $\mu(G_l) = l^{-1}\sigma_l\mu(G)$. By Lemma 4.1, we have $\mu(G_l^{\otimes d}) = l^{-d}\sigma_{l^d}\mu(G_1^{\otimes d})$. Thus,

$$\mu(X \otimes G_l^{\otimes d}) \stackrel{(4.1)}{=} \mu(X \otimes l^{-d} \sigma_{l^d} \mu(G_1^{\otimes d})) \stackrel{L.4.1}{=} l^{-d} \sigma_{l^d} \mu(X \otimes G_1^{\otimes d}).$$

Therefore, we have

$$\|X\otimes G_l^{\otimes d}\|_{1,\infty} = \sup_{t>0} \frac{t}{l^d} \mu(\frac{t}{l^d}, X\otimes G_1^{\otimes d}) = \sup_{s>0} s\mu(s, X\otimes G_1^{\otimes d}) = \|X\otimes G_1^{\otimes d}\|_{1,\infty}.$$

Clearly, $\mu(G_1) \geq \frac{1}{e\sqrt{\pi}}\chi_{(0,1)}$. It follows that

$$\|X \otimes G_1^{\otimes d}\|_{1,\infty} \stackrel{(4.1)}{=} \|X \otimes \mu(G_1)^{\otimes d}\|_{1,\infty} \ge \|X \otimes (\frac{1}{e\sqrt{\pi}}\chi_{(0,1)})^{\otimes d}\|_{1,\infty} = e^{-d}\pi^{-\frac{d}{2}} \|X\|_{1,\infty}.$$

This proves the left hand side inequality.

The following lemma is ideologically similar to Theorem II.4.3 in [28].

Lemma 4.3. If g is a smooth homogeneous function on $\mathbb{R}^2 \setminus \{0\}$, then $\mathcal{F}g$ satisfies (possibly, after some δ distribution is subtracted) the conditions (2.2).

Proof. By assumption, g is a smooth function on the circle $\{|z|=1\}$. Thus,

$$g(e^{i\theta}) = \sum_{k \in \mathbb{Z}} \alpha_k e^{ik\theta},$$

³Without loss of generality, \mathcal{M} is atomless. Suppose first that $x \in \mathcal{M}$ is τ -compact. By Theorem 2.3.11 in [18], there exists a trace preserving *-isomorphism $i:L_{\infty}(0,\infty) \to \mathcal{M}_1$ such that $i_1(\mu(A)) = |A|$. Consider trace preserving isomorphism $i \otimes 1:L_{\infty}(0,\infty) \otimes L_{\infty}(0,\infty) \to \mathcal{M} \otimes L_{\infty}(0,\infty)$. We have $i(\mu(A) \otimes g) = |A| \otimes g$. Since every trace preserving *-isomorphism preserves singular value function, the claim follows for τ -compact operators. The general case follows by approximation.

where Fourier coefficients decrease faster than every power. Therefore,

$$g = \sum_{k \in \mathbb{Z}} \alpha_k g_k, \quad g_k(z) = \frac{z^k}{|z|^k}, \quad 0 \neq z \in \mathbb{C}.$$

For every $k \neq 0$, we have⁴

$$(\mathcal{F}g_k)(z) = \frac{|k|}{2\pi i^k} \cdot \frac{g_k(z)}{|z|^2}, \quad 0 \neq z \in \mathbb{C}.$$

Hence,

$$(\mathcal{F}g)(z) = \alpha_0 \delta + \frac{1}{|z|^2} h(e^{i\operatorname{Arg}(z)}),$$

where the smooth function h on the circle is defined by the formula

$$h(e^{i\theta}) = \sum_{0 \neq k \in \mathbb{Z}} \frac{|k|}{2\pi i^k} \alpha_k e^{ik\theta}.$$

So, $(\mathcal{F}g - \alpha_0 \delta)(z) = O(|z|^{-2})$. Furthermore, have

$$\nabla(\frac{h(e^{i\operatorname{Arg}(z)})}{|z|^2}) = h(e^{i\operatorname{Arg}(z)}) \cdot \nabla(\frac{1}{|z|^2}) + \frac{1}{|z|^2} \cdot \frac{dh(e^{i\theta})}{d\theta}|_{\theta = \operatorname{Arg}(z)} \cdot \nabla(\operatorname{Arg}(z)) = O(\frac{1}{|z|^3}).$$

This completes the verification that $\mathcal{F}g - \alpha_0 \delta$ satisfies condition (2.2).

Theorem 4.4. For every $A = A^* \in \mathcal{M}$ with $\operatorname{spec}(A) \subset \mathbb{Z}$ and for every Lipschitz function f, we have

$$||T_{f^{[1]}}^{A,A}(V)||_{1,\infty} \le c_{abs}||f'||_{\infty}||V||_{1}, \quad V \in L_{1}(\mathcal{M}).$$

Proof. Fix a smooth homogeneous function g on \mathbb{R}^2 such that $g(e^{i\theta}) = \tan(\theta)$ for $\theta \in (-\frac{\pi}{4}, \frac{\pi}{4})$ and for $\theta \in (\frac{3\pi}{4}, \frac{5\pi}{4})$. Without loss of generality, g is mean zero on the circle $\{|z|=1\}$. By Lemma 4.3, $\mathcal{F}g$ satisfies the conditions (2.2). The operator $g(\nabla) \in B(L_2(\mathbb{R}^2))$ since g is bounded. Recall that $(g(\nabla))(x) = (\mathcal{F}g) * x$. By Theorem 2.1, we have

$$1 \otimes g(\nabla) : L_1(\mathcal{M} \otimes L_{\infty}(\mathbb{R}^2)) \to L_{1,\infty}(\mathcal{M} \otimes L_{\infty}(\mathbb{R}^2)).$$

Consider Schwartz functions⁵ ϕ_m on \mathbb{R}^2 which vanish near 0, such that $\phi_m(t) = 1$ for $|t| \in (\frac{1}{m}, m)$ and such that $\|\mathcal{F}\phi_m\|_1 \le c_{abs}$ for all $m \ge 1$. It follows that

$$||1\otimes (g\phi_m)(\nabla)||_{L_1\to L_{1,\infty}}\leq ||1\otimes g(\nabla)||_{L_1\to L_{1,\infty}}||1\otimes \phi_m(\nabla)||_{L_1\to L_1}\leq$$

$$\leq \|1 \otimes g(\nabla)\|_{L_1 \to L_{1,\infty}} \|\mathcal{F}\phi_m\|_1 \leq c_{abs} \|1 \otimes g(\nabla)\|_{L_1 \to L_{1,\infty}} = c_{abs}, \quad m \geq 1.$$

The last equality holds because g is fixed.

$$\mathcal{F}(\psi_m) = \mathcal{F}(\sigma_m \psi) * \mathcal{F}(1 - \sigma_{\frac{1}{m}} \psi) = m \sigma_{\frac{1}{m}} (\mathcal{F}(\psi)) - m \sigma_{\frac{1}{m}} (\mathcal{F}(\psi)) * \frac{1}{m} \sigma_m (\mathcal{F}(\psi)).$$

Applying Young's inequality, we conclude that

$$\|\mathcal{F}(\psi_m)\|_1 \leq \|m\sigma_{\frac{1}{m}}(\mathcal{F}(\psi))\|_1 + \|m\sigma_{\frac{1}{m}}(\mathcal{F}(\psi))\|_1 \|\frac{1}{m}\sigma_m(\mathcal{F}(\psi))\|_1 = \|\mathcal{F}(\psi)\|_1 + \|\mathcal{F}(\psi)\|_1^2$$

Consider the functions $\phi_m = \psi_{3m}^{\otimes 2}$. By Fubini Theorem, $\sup_{m \geq 1} \|\mathcal{F}(\phi_m)\|_1 < \infty$. Clearly, $\psi_m = 1$ on the set $[-m,m] \setminus [-\frac{2}{m},\frac{2}{m}]$. Thus, $\phi_m(t) = 1$ if $t \in 3mK$ and $3mt \notin 2K$, where $K = [-1,1] \times [-1,1]$. Thus, $\phi_m(t) = 1$ whenever $|t| \in (\frac{1}{m},m)$.

⁴This can be checked e.g. by substituting $m=\Omega=g_k$ into the formula (26) in Theorem II.4.3 in [28].

⁵Let ψ be a Schwartz function on \mathbb{R} which is 1 on (-1,1) and which is supported on (-2,2). Set $\psi_m = \sigma_m \psi \cdot (1 - \sigma_{\frac{1}{n}} \psi)$. It follows that

By assumption, $A = \sum_{j \in \mathbb{Z}} j p_j$, where $\{p_j\}_{j \in \mathbb{Z}}$ are pairwise orthogonal projections such that $\sum_{j \in \mathbb{Z}} p_j = 1$. Since A is bounded, it follows that $p_j = 0$ for all but finitely many $j \in \mathbb{Z}$. Hence, sums are, in fact, finite. Consider a unitary operator

$$u = \sum_{j \in \mathbb{Z}} p_j \otimes e_{(j, f(j))},$$

where $e_{(j,f(j))}$ is given in (3.4). Without loss of generality, $||f'||_{\infty} \leq 1$. For every $m \geq ||A||_{\infty}$, we have $|i-j|, |f(i)-f(j)| \leq 2m$ for every $i, j \in \operatorname{spec}(A)$. Hence,

$$(g\phi_m)(i-j, f(i)-f(j)) = g(i-j, f(i)-f(j)) = \frac{f(i)-f(j)}{i-j}, \quad i, j \in \operatorname{spec}(A), \quad i \neq j.$$

It follows from the preceding paragraph and from the equality $\|G_I^{\otimes 2}\|_1 = 1$ that

$$(4.2) \|(1 \otimes (g\phi_m)(\nabla))(u(V \otimes G_l^{\otimes 2})u^*)\|_{1,\infty} \le c_{abs} \|u(V \otimes G_l^{\otimes 2})u^*\|_1 = c_{abs} \|V\|_1.$$

It is clear that

$$(1 \otimes (g\phi_m)(\nabla))(u(V \otimes G_l^{\otimes 2})u^*) = \sum_{i,j} p_i V p_j \otimes (g\phi_m(\nabla))(G_l^{\otimes 2} e_{(i-j,f(i)-f(j))}).$$

Since there are only finitely many summands, it follows from Lemma 3.3 (as applied to the Schwartz function $g\phi_m$) that

$$(1 \otimes (g\phi_m)(\nabla))(u(V \otimes G_l^{\otimes 2})u^*) - \sum_{i \neq j} p_i V p_j \otimes \frac{f(i) - f(j)}{i - j} G_l^{\otimes 2} e_{(i - j, f(i) - f(j))} \to 0$$

in $L_1(\mathcal{M} \otimes L_\infty(\mathbb{R}^2))$ as $l \to \infty$. It is immediate that

$$\sum_{i \neq j} p_i V p_j \otimes \frac{f(i) - f(j)}{i - j} G_l^{\otimes 2} e_{(i - j, f(i) - f(j))} =$$

$$= \left(\sum_{k \in \mathbb{Z}} p_k \otimes e_{(k,f(k))}\right) \cdot \left(\sum_{i \neq j} p_i V p_j \otimes \frac{f(i) - f(j)}{i - j} G_l^{\otimes 2}\right) \cdot \left(\sum_{l \in \mathbb{Z}} p_l \otimes e_{(-l, -f(l))}\right) =$$

$$\stackrel{(2.5)}{=} u(T_{f^{[1]}}^{A,A}(V) \otimes G_l^{\otimes 2}) u^*.$$

Therefore,

$$(4.3) (1 \otimes (g\phi_m)(\nabla))(u(V \otimes G_l^{\otimes 2})u^*) - u(T_{f^{[1]}}^{A,A}(V) \otimes G_l^{\otimes 2})u^* \to 0$$

in $L_1(\mathcal{M} \otimes L_{\infty}(\mathbb{R}^2))$ (and, hence, in $L_{1,\infty}(\mathcal{M} \otimes L_{\infty}(\mathbb{R}^2))$) as $l \to \infty$. Combining (4.2) and (4.3), we arrive at

$$\limsup_{l \to \infty} \|u(T_{f^{[1]}}^{A,A}(V) \otimes G_l^{\otimes 2})u^*\|_{1,\infty} \le c_{abs}\|V\|_1.$$

Since u is unitary, it follows that

$$\limsup_{l \to \infty} \|T_{f^{[1]}}^{A,A}(V) \otimes G_{l}^{\otimes 2}\|_{1,\infty} \le c_{abs} \|V\|_{1}.$$

The assertion follows now from Lemma 4.2.

5. Proof of the main results

In this section we collect the results announced in the abstract and its corollaries. Throughout this section fix a semi-finite von Neumann algebra \mathcal{M} with normal, semi-finite, faithful trace τ .

Lemma 5.1. Let $A = A^* \in \mathcal{M}$. If $\{\xi_n\}_{n\geq 0}$ is a uniformly bounded sequence of Borel functions on \mathbb{R}^2 such that $\xi_n \to \xi$ everywhere, then

(5.1)
$$T_{\xi_n}^{A,A}(V) \to T_{\xi}^{A,A}(V), \quad V \in L_2(\mathcal{M})$$

in $L_2(\mathcal{M})$ as $n \to \infty$.

Proof. Let ν be a projection valued measure on \mathbb{R}^2 considered in Subsection 2.4 (see (2.4)). Let $\gamma: \mathbb{R} \to \mathbb{R}^2$ be a Borel measurable bijection. Clearly, $\nu \circ \gamma$ is a projection valued measure on \mathbb{R} . Hence, there exists a self-adjoint operator B acting on the Hilbert space $L_2(\mathcal{M})$ such that $E_B = \nu \circ \gamma$.

Set $\eta_n = \xi_n \circ \gamma$ and $\eta = \xi \circ \gamma$. We have $\eta_n \to \eta$ everywhere on \mathbb{R} . Thus,

$$T_{\xi_n}^{A,A} = \int_{\mathbb{R}^2} \xi_n d\nu = \int_{\mathbb{R}} \eta_n(\lambda) dE_B(\lambda) = \eta_n(B) \to \eta(B) =$$
$$= \int_{\mathbb{R}} \eta(\lambda) dE_B(\lambda) = \int_{\mathbb{R}^2} \xi d\nu = T_{\xi}^{A,A}.$$

Here, the convergence is understood with respect to the strong operator topology on the space $B(L_2(\mathcal{M}))$. In particular, (5.1) follows.

Proof of Theorem 1.2. Step 1. Let A is bounded. For every $n \geq 1$, set

$$A_n \stackrel{def}{=} \sum_{k \in \mathbb{Z}} \frac{k}{n} E_A([\frac{k}{n}, \frac{k+1}{n})),$$

$$\xi_n(t,s) = f^{[1]}(\frac{k}{n}, \frac{l}{n}), \quad t \in [\frac{k}{n}, \frac{k+1}{n}), s \in [\frac{l}{n}, \frac{l+1}{n}).$$

It is immediate that (see e.g. Lemma 8 in [25] for much stronger assertion)

$$T_{\xi_n}^{A,A}(V) = T_{f^{[1]}}^{A_n,A_n}(V) = T_{(n\sigma_n f)^{[1]}}^{nA_n,nA_n}(V).$$

It follows from Theorem 4.4 that

$$||T_{\xi_n}^{A,A}(V)||_{1,\infty} \le c_{abs}||(n\sigma_n f)'||_{\infty}||V||_1 = c_{abs}||f'||_{\infty}||V||_1.$$

Note that $\xi_n \to f^{[1]}$ everywhere. It follows from Lemma 5.1 that

$$T_{\xi_n}^{A,A}(V) \to T_{f^{[1]}}(V), \quad V \in L_2(\mathcal{M})$$

in $L_2(\mathcal{M})$ (and, hence, in measure — see e.g [20]) as $n \to \infty$. Since the quasi-norm in $L_{1,\infty}(\mathcal{M})$ is a Fatou quasi-norm, it follows that

$$||T_{f^{[1]}}^{A,A}(V)||_{1,\infty} \le c_{abs}||f'||_{\infty}||V||_{1}, \quad V \in (L_{1} \cap L_{2})(\mathcal{M}).$$

Step 2. Let now A be an arbitrary operator affiliated with \mathcal{M} . Set $A_n = AE_A([-n,n])$. By Step 1, we have

$$||T_{f^{[1]}}^{A_n,A_n}(V)||_{1,\infty} \le c_{abs}||f'||_{\infty}||V||_1.$$

It follows immediately from the definition of the double operator integral that

$$T_{f^{[1]}}^{A_n,A_n}(V) = E_A([-n,n]) \cdot T_{f^{[1]}}^{A,A}(V) \cdot E_A([-n,n]) \to T_{f^{[1]}}^{A,A}(V)$$

in $L_2(\mathcal{M})$ (and, hence, in measure) as $n \to \infty$. Since the quasi-norm in $L_{1,\infty}(\mathcal{M})$ is a Fatou quasi-norm, the assertion follows.

The following lemma is ideologically similar to Theorem 7.4 in [20].

Lemma 5.2. If $A, B \in \mathcal{M}$ are such that $[A, B] \in L_2(\mathcal{M})$, then, for every Lipschitz function f, we have

$$T_{f^{[1]}}^{A,A}([A,B]) = [f(A),B].$$

Proof. By definition of double operator integral given in Subsection 2.4, we have

$$(5.2) T_{\xi_1}^{A,A} T_{\xi_2}^{A,A} = T_{\xi_1 \xi_2}^{A,A}.$$

Let $\xi_1 = f^{[1]}$ and let $\xi_2(\lambda, \mu) = \lambda - \mu$ when $|\lambda|, |\mu| \leq ||A||_{\infty}$. $\xi_2(\lambda, \mu) = 0$ when $|\lambda| > ||A||_{\infty}$ or $|\mu| \leq ||A||_{\infty}$.

If p is a τ -finite projection, then $pB \in L_2(\mathcal{M})$ and

$$T_{\xi_1 \xi_2}^{A,A}(pB) = f(A)pB - pBf(A), \quad T_{\xi_2}^{A,A}(pB) = ApB - pBA,$$

Applying (5.2) to the operator $pB \in L_2(\mathcal{M})$, we obtain

(5.3)
$$T_{f^{[1]}}^{A,A}(ApB - pBA) = f(A)pB - pBf(A).$$

Applying Proposition 6.6 in [20] to the operator nA, we construct a sequence $\{p_{n,k}\}_{k\geq 0}$ of τ -finite projections such that $p_{n,k}\uparrow 1$ as $k\to\infty$ and such that $\|[nA,p_{n,k}]\|_2\leq 1$. Let $\{\eta_m\}_{m\geq 0}$ be an orthonormal basis in $L_2(\mathcal{M})$. Fix k_n so large that

(5.4)
$$||(1 - p_{n,k_n})\eta_m||_2 \le \frac{1}{n}, \quad 0 \le m < n.$$

Set $q_n = p_{n,k_n}$. It follows from (5.4) that $q_n \to 1$ in the strong operator topology (in the left regular representation of \mathcal{M}). Clearly, $[A, q_n] \to 0$ in $L_2(\mathcal{M})$. By construction,

$$Aq_nB - q_nBA = [A, q_n]B + q_n[A, B] \rightarrow [A, B], \quad n \rightarrow \infty,$$

in $L_2(\mathcal{M})$. Since $T_{f^{[1]}}^{A,A}$ is bounded, it follows that

$$f(A)q_nB - q_nBf(A) \stackrel{(5.3)}{=} T_{f^{[1]}}^{A,A}(Aq_nB - q_nBA) \to T_{f^{[1]}}^{A,A}(AB - BA), \quad n \to \infty,$$

in $L_2(\mathcal{M})$. On the other hand,

$$f(A)q_nB - q_nBf(A) \to [f(A), B], \quad n \to \infty,$$

in the strong operator topology (in the left regular representation of \mathcal{M}). This concludes the proof.

Theorem 5.3. For all self-adjoint operators $A, B \in \mathcal{M}$ such that $[A, B] \in L_1(\mathcal{M})$ and for every Lipschitz function f, we have

$$||[f(A), B]||_{1,\infty} \le c_{abs} ||f'||_{\infty} \cdot ||[A, B]||_{1}.$$

For all self-adjoint operators $X, Y \in \mathcal{M}$ such that $X - Y \in L_1(\mathcal{M})$ and for every Lipschitz function f, we have

$$||f(X) - f(Y)||_{1,\infty} \le c_{abs} ||f'||_{\infty} ||X - Y||_1.$$

Proof. By assumption, $[A, B] \in (L_1 \cap L_2)(\mathcal{M})$. The first assertion follows by combining Lemma 5.2 and Theorem 1.2. Applying the first assertion to the operators

$$A = \begin{pmatrix} X & 0 \\ 0 & Y \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

we obtain the second assertion.

References

- M. S. Birman and M. Z. Solomyak, Double Stieltjes operator integrals (Russian), Probl. Math. Phys., Izdat. Leningrad. Univ., Leningrad, (1966) 33-67. English translation in: Topics in Mathematical Physics, Vol. 1 (1967), Consultants Bureau Plenum Publishing Corporation, New York, 25-54.
- [2] Birman M., Solomyak M. Spectral theory of selfadjoint operators in Hilbert space. Mathematics and its Applications (Soviet Series). D. Reidel Publishing Co., Dordrecht, 1987.
- [3] Caspers M., Montgomery-Smith S., Potapov D., Sukochev F. The best constants for operator Lipschitz functions on Schatten classes. J. Funct. Anal. 267 (2014), no. 10, 3557–3579.
- [4] Caspers M., Potapov D., Sukochev F., Zanin D. Weak type estimates for the absolute value mapping. J. Operator Theory, 73 (2015), no. 2, 101–124.
- [5] Davies E. Lipschitz continuity of functions of operators in the Schatten classes. J. Lond. Math. Soc. 37 (1988) 148-157.
- [6] de Leeuw, K. On L_p multipliers. Ann. of Math. (2) 81 (1965) 364–379.
- [7] Dodds P., Dodds T., de Pagter B., Sukochev F. Lipschitz continuity of the absolute value and Riesz projections in symmetric operator spaces. J. Funct. Anal. 148 (1997), 28–69.
- [8] Dodds P., Dodds T., de Pagter B., Sukochev F. Lipschitz continuity of the absolute value in preduals of semifinite factors. Integral Equations Operator Theory 34 (1999), 28–44.
- [9] Farforovskaya Y. An estimate of the nearness of the spectral decompositions of self-adjoint operators in the Kantorovich-Rubinstein metric. Vestnik Leningrad. Univ. 22 (1967), no. 19, 155–156.
- [10] Farforovskaya Y. The connection of the Kantorovich-Rubinstein metric for spectral resolutions of selfadjoint operators with functions of operators. Vestnik Leningrad. Univ. 23 (1968), no. 19, 94–97.
- [11] Farforovskaya Y. An example of a Lipschitz function of self-adjoint operators with non-nuclear difference under a nuclear perturbation. Zap. Nauchn. Sem. Leningrad. Otdel. Mat. Inst. Steklov. (LOMI) 30 (1972), 146–153.
- [12] Grafakos L. Classical Fourier analysis. Third edition. Graduate Texts in Mathematics, 249. Springer, New York, 2014.
- [13] Halmos P., von Neumann J. Operator methods in classical mechanics. II. Ann. of Math. (2) 43, (1942). 332–350.
- [14] Kalton N., Sukochev F. Symmetric norms and spaces of operators. J. Reine Angew. Math. 621 (2008), 81–121.
- [15] Kato T. Continuity of the map $S \mapsto |S|$ for linear operators. Proc. Japan Acad. **49** (1973) 157–160.
- [16] Kosaki H. Unitarily invariant norms under which the map $A\mapsto |A|$ is continuous. Publ. Res. Inst. Math. Sci. 28 (1992), 299–313.
- [17] Krein M. Some new studies in the theory of perturbations of self-adjoint operators. First Math. Summer School, Part I (Russian), Izdat. "Naukova Dumka", Kiev, 1964, pp. 103–187.
- [18] Lord S., Sukochev F., Zanin D. Singular traces. Theory and applications. De Gruyter Studies in Mathematics, 46. De Gruyter, Berlin, 2013.
- [19] Nazarov F., Peller V. Lipschitz functions of perturbed operators. C. R. Math. Acad. Sci. Paris 347 (2009), 857–862.
- [20] de Pagter B., Witvliet H., Sukochev F. Double operator integrals. J. Funct. Anal. 192 (2002), no. 1, 52–111.
- [21] Parcet J. Pseudo-localization of singular integrals and noncommutative Calderón-Zygmund theory. J. Funct. Anal. 256 (2009), 509–593.
- [22] Peller V. Hankel operators in the theory of perturbations of unitary and selfadjoint operators. Funktsional. Anal. i Prilozhen. 19 (1985), no. 2, 37–51, 96.

- [23] Potapov D., Sukochev F. Lipschitz and commutator estimates in symmetric operator spaces. J. Operator Theory 59 (2008) 211–234.
- [24] Potapov D., Sukochev F. Operator-Lipschitz functions in Schatten-von Neumann classes. Acta Math. 207 (2011), no. 2, 375–389.
- [25] Potapov D., Sukochev F. Unbounded Fredholm modules and double operator integrals. J. Reine Angew. Math. 626 (2009), 159–185.
- [26] Randrianantoanina N. A weak type inequality for non-commutative martingales and applications. Proc. London Math. Soc. (3) 91 (2005), no. 2, 509–542.
- [27] de la Salle M. A shorter proof of a result by Potapov and Sukochev on Lipschitz functions on S_p . arXiv:0905.1055.
- [28] Stein E. Singular integrals and differentiability properties of functions. Princeton Mathematical Series, No. 30 Princeton University Press, Princeton, N.J.
- [29] Sukochev F. On a conjecture of A. Bikchentaev. Proc. Sympos. Pure Math., 87, Amer. Math. Soc., Providence, R.I., 2013.
- [30] Sukochev F. Completeness of quasi-normed symmetric operator spaces. Indag. Math. (N.S.) **25** (2014), no. 2, 376–388.
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